

Special Instructions / Useful Data

$\mathbb{Z}_n = \{\overline{0}, \overline{1}, \dots, \overline{n-1}\}$ denotes the additive group of integers modulo n

\mathbb{R} = the set of all real numbers

\mathbb{N} = the set of all positive integers

\mathbb{Z} = the set of all integers

\mathbb{C} = the set of all complex numbers

\mathbb{Q} = the set of all rational numbers

$\gcd(r, n)$ = the greatest common divisor of the integers r and n

S_n = the symmetric group of all permutations of $\{1, 2, \dots, n\}$

A_n = the group of all even permutations in S_n

$M_n(\mathbb{C})$ = the set of all $n \times n$ matrices with entries from \mathbb{C}

$M_n(\mathbb{R})$ = the set of all $n \times n$ matrices with entries from \mathbb{R}

M^T = the transpose of the matrix M

I_n = the $n \times n$ identity matrix

$P_n(x)$ = the real vector space of polynomials, in the variable x with real coefficients and having degree at most n , together with the zero polynomial. These polynomials are regarded as functions from \mathbb{R} to \mathbb{R}

$\binom{n}{k}$ = the binomial coefficient defined as $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

$f \circ g$ = the composite function defined by $(f \circ g)(x) = f(g(x))$

$A \setminus B$ = the complement of the set B in the set A , that is, $\{x \in A : x \notin B\}$

$\log x$ = the logarithm of x to the base e for a positive number x

\mathbb{R}^n = the n -dimensional Euclidean space

$A \times B$ = the Cartesian product of the sets A and B

M^{-1} = the inverse of an invertible matrix M

Section A: Q.1 – Q.10 Carry ONE mark each.

Q.1 Let $y_c: \mathbb{R} \rightarrow (0, \infty)$ be the solution of the Bernoulli's equation

$$\frac{dy}{dx} - y + y^3 = 0, \quad y(0) = c > 0.$$

Then, for every $c > 0$, which one of the following is true?

- (A) $\lim_{x \rightarrow \infty} y_c(x) = 0$
- (B) $\lim_{x \rightarrow \infty} y_c(x) = 1$
- (C) $\lim_{x \rightarrow \infty} y_c(x) = e$
- (D) $\lim_{x \rightarrow \infty} y_c(x)$ does not exist

Q.2 For a twice continuously differentiable function $g: \mathbb{R} \rightarrow \mathbb{R}$, define

$$u_g(x, y) = \frac{1}{y} \int_{-y}^y g(x+t) dt \quad \text{for } (x, y) \in \mathbb{R}^2, \quad y > 0.$$

Which one of the following holds for all such g ?

(A) $\frac{\partial^2 u_g}{\partial x^2} = \frac{2}{y} \frac{\partial u_g}{\partial y} + \frac{\partial^2 u_g}{\partial y^2}$

(B) $\frac{\partial^2 u_g}{\partial x^2} = \frac{1}{y} \frac{\partial u_g}{\partial y} + \frac{\partial^2 u_g}{\partial y^2}$

(C) $\frac{\partial^2 u_g}{\partial x^2} = \frac{2}{y} \frac{\partial u_g}{\partial y} - \frac{\partial^2 u_g}{\partial y^2}$

(D) $\frac{\partial^2 u_g}{\partial x^2} = \frac{1}{y} \frac{\partial u_g}{\partial y} - \frac{\partial^2 u_g}{\partial y^2}$

Q.3 Let $y(x)$ be the solution of the differential equation

$$\frac{dy}{dx} = 1 + y \sec x \quad \text{for } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

that satisfies $y(0) = 0$. Then, the value of $y\left(\frac{\pi}{6}\right)$ equals

(A) $\sqrt{3} \log\left(\frac{3}{2}\right)$

(B) $\left(\frac{\sqrt{3}}{2}\right) \log\left(\frac{3}{2}\right)$

(C) $\left(\frac{\sqrt{3}}{2}\right) \log 3$

(D) $\sqrt{3} \log 3$

Q.4 Let \mathcal{F} be the family of curves given by

$$x^2 + 2hxy + y^2 = 1, \quad -1 < h < 1.$$

Then, the differential equation for the family of orthogonal trajectories to \mathcal{F} is

(A) $(x^2y - y^3 + y) \frac{dy}{dx} - (xy^2 - x^3 + x) = 0$

(B) $(x^2y - y^3 + y) \frac{dy}{dx} + (xy^2 - x^3 + x) = 0$

(C) $(x^2y + y^3 + y) \frac{dy}{dx} - (xy^2 + x^3 + x) = 0$

(D) $(x^2y + y^3 + y) \frac{dy}{dx} + (xy^2 + x^3 + x) = 0$

Q.5 Let G be a group of order 39 such that it has exactly one subgroup of order 3 and exactly one subgroup of order 13. Then, which one of the following statements is TRUE?

(A) G is necessarily cyclic

(B) G is abelian but need not be cyclic

(C) G need not be abelian

(D) G has 13 elements of order 13

Q.6 For a positive integer n , let $U(n) = \{\bar{r} \in \mathbb{Z}_n : \gcd(r, n) = 1\}$ be the group under multiplication modulo n . Then, which one of the following statements is TRUE?

- (A) $U(5)$ is isomorphic to $U(8)$
- (B) $U(10)$ is isomorphic to $U(12)$
- (C) $U(8)$ is isomorphic to $U(10)$
- (D) $U(8)$ is isomorphic to $U(12)$

Q.7 Which one of the following is TRUE for the symmetric group S_{13} ?

- (A) S_{13} has an element of order 42
- (B) S_{13} has no element of order 35
- (C) S_{13} has an element of order 27
- (D) S_{13} has no element of order 60

Q.8 Let G be a finite group containing a non-identity element which is conjugate to its inverse. Then, which one of the following is TRUE?

- (A) The order of G is necessarily even
- (B) The order of G is not necessarily even
- (C) G is necessarily cyclic
- (D) G is necessarily abelian but need not be cyclic

Q.9 Consider the following statements.

P: If a system of linear equations $Ax = b$ has a unique solution, where A is an $m \times n$ matrix and b is an $m \times 1$ matrix, then $m = n$.

Q: For a subspace W of a nonzero vector space V , whenever $u \in V \setminus W$ and $v \in V \setminus W$, then $u + v \in V \setminus W$.

Which one of the following holds?

- (A) Both P and Q are true
- (B) P is true but Q is false
- (C) P is false but Q is true
- (D) Both P and Q are false

Q.10 Let $g: \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. Which one of the following is the solution of the differential equation

$$\frac{d^2y}{dx^2} + y = g(x) \text{ for } x \in \mathbb{R},$$

satisfying the conditions $y(0) = 0$, $y'(0) = 1$?

(A) $y(x) = \sin x - \int_0^x \sin(x-t) g(t) dt$

(B) $y(x) = \sin x + \int_0^x \sin(x-t) g(t) dt$

(C) $y(x) = \sin x - \int_0^x \cos(x-t) g(t) dt$

(D) $y(x) = \sin x + \int_0^x \cos(x-t) g(t) dt$

Section A: Q.11 – Q.30 Carry TWO marks each.

Q.11 Which one of the following groups has elements of order 1, 2, 3, 4, 5 but does not have an element of order greater than or equal to 6 ?

(A) The alternating group A_6

(B) The alternating group A_5

(C) S_6

(D) S_5

Q.12 Consider the group $G = \{A \in M_2(\mathbb{R}) : AA^T = I_2\}$ with respect to matrix multiplication. Let

$$Z(G) = \{A \in G : AB = BA, \text{ for all } B \in G\}.$$

Then, the cardinality of $Z(G)$ is

(A) 1

(B) 2

(C) 4

(D) Infinite

Q.13 Let V be a nonzero subspace of the complex vector space $M_7(\mathbb{C})$ such that every nonzero matrix in V is invertible. Then, the dimension of V over \mathbb{C} is

- (A) 1
- (B) 2
- (C) 7
- (D) 49

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Q.14 For $n \in \mathbb{N}$, let

$$a_n = \frac{1}{(3n+2)(3n+4)} \quad \text{and} \quad b_n = \frac{n^3 + \cos(3^n)}{3^n + n^3}.$$

Then, which one of the following is TRUE?

(A) $\sum_{n=1}^{\infty} a_n$ is convergent but $\sum_{n=1}^{\infty} b_n$ is divergent

(B) $\sum_{n=1}^{\infty} a_n$ is divergent but $\sum_{n=1}^{\infty} b_n$ is convergent

(C) Both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are divergent

(D) Both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent

Q.15

Let $a = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} \\ 0 \end{bmatrix}$. Consider the following two statements.

P: The matrix $I_4 - aa^T$ is invertible.

Q: The matrix $I_4 - 2aa^T$ is invertible.

Then, which one of the following holds?

- (A) P is false but Q is true
- (B) P is true but Q is false
- (C) Both P and Q are true
- (D) Both P and Q are false

Q.16 Let A be a 6×5 matrix with entries in \mathbb{R} and B be a 5×4 matrix with entries in \mathbb{R} . Consider the following two statements.

P: For all such nonzero matrices A and B , there is a nonzero matrix Z such that AZB is the 6×4 zero matrix.

Q: For all such nonzero matrices A and B , there is a nonzero matrix Y such that BYA is the 5×5 zero matrix.

Which one of the following holds?

- (A) Both P and Q are true
- (B) P is true but Q is false
- (C) P is false but Q is true
- (D) Both P and Q are false

Q.17 Let $P_{11}(x)$ be the real vector space of polynomials, in the variable x with real coefficients and having degree at most 11, together with the zero polynomial.

Let

$$E = \{s_0(x), s_1(x), \dots, s_{11}(x)\}, \quad F = \{r_0(x), r_1(x), \dots, r_{11}(x)\}$$

be subsets of $P_{11}(x)$ having 12 elements each and satisfying

$$s_0(3) = s_1(3) = \dots = s_{11}(3) = 0, \quad r_0(4) = r_1(4) = \dots = r_{11}(4) = 1.$$

Then, which one of the following is TRUE?

- (A) Any such E is not necessarily linearly dependent and any such F is not necessarily linearly dependent
- (B) Any such E is necessarily linearly dependent but any such F is not necessarily linearly dependent
- (C) Any such E is not necessarily linearly dependent but any such F is necessarily linearly dependent
- (D) Any such E is necessarily linearly dependent and any such F is necessarily linearly dependent

Q.18 For the differential equation

$$y(8x - 9y)dx + 2x(x - 3y)dy = 0,$$

which one of the following statements is TRUE?

- (A) The differential equation is not exact and has x^2 as an integrating factor
- (B) The differential equation is exact and homogeneous
- (C) The differential equation is not exact and does not have x^2 as an integrating factor
- (D) The differential equation is not homogeneous and has x^2 as an integrating factor

Q.19 For $x \in \mathbb{R}$, let $[x]$ denote the greatest integer less than or equal to x .

For $x, y \in \mathbb{R}$, define

$$\min\{x, y\} = \begin{cases} x & \text{if } x \leq y, \\ y & \text{otherwise.} \end{cases}$$

Let $f: [-2\pi, 2\pi] \rightarrow \mathbb{R}$ be defined by

$$f(x) = \sin(\min\{x, x - [x]\}) \text{ for } x \in [-2\pi, 2\pi].$$

Consider the set $S = \{x \in [-2\pi, 2\pi]: f \text{ is discontinuous at } x\}$.

Which one of the following statements is TRUE?

- (A) S has 13 elements
- (B) S has 7 elements
- (C) S is an infinite set
- (D) S has 6 elements

Q.20 Define the sequences $\{a_n\}_{n=3}^{\infty}$ and $\{b_n\}_{n=3}^{\infty}$ as

$$a_n = (\log n + \log \log n)^{\log n} \quad \text{and} \quad b_n = n^{\left(1 + \frac{1}{\log n}\right)}.$$

Which one of the following is TRUE?

(A) $\sum_{n=3}^{\infty} \frac{1}{a_n}$ is convergent but $\sum_{n=3}^{\infty} \frac{1}{b_n}$ is divergent

(B) $\sum_{n=3}^{\infty} \frac{1}{a_n}$ is divergent but $\sum_{n=3}^{\infty} \frac{1}{b_n}$ is convergent

(C) Both $\sum_{n=3}^{\infty} \frac{1}{a_n}$ and $\sum_{n=3}^{\infty} \frac{1}{b_n}$ are divergent

(D) Both $\sum_{n=3}^{\infty} \frac{1}{a_n}$ and $\sum_{n=3}^{\infty} \frac{1}{b_n}$ are convergent

Q.21 For $p, q, r \in \mathbb{R}$, $r \neq 0$ and $n \in \mathbb{N}$, let

$$a_n = p^n n^q \left(\frac{n}{n+2} \right)^{n^2} \quad \text{and} \quad b_n = \frac{n^n}{n! r^n} \left(\sqrt{\frac{n+2}{n}} \right).$$

Then, which one of the following statements is TRUE?

(A) If $1 < p < e^2$ and $q > 1$, then $\sum_{n=1}^{\infty} a_n$ is convergent

(B) If $e^2 < p < e^4$ and $q > 1$, then $\sum_{n=1}^{\infty} a_n$ is convergent

(C) If $1 < r < e$, then $\sum_{n=1}^{\infty} b_n$ is convergent

(D) If $\frac{1}{e} < r < 1$, then $\sum_{n=1}^{\infty} b_n$ is convergent

- Q.22 Let $P_7(x)$ be the real vector space of polynomials, in the variable x with real coefficients and having degree at most 7, together with the zero polynomial. Let $T: P_7(x) \rightarrow P_7(x)$ be the linear transformation defined by

$$T(f(x)) = f(x) + \frac{df(x)}{dx}.$$

Then, which one of the following is TRUE?

- (A) T is not a surjective linear transformation
- (B) There exists $k \in \mathbb{N}$ such that T^k is the zero linear transformation
- (C) 1 and 2 are the eigenvalues of T
- (D) There exists $r \in \mathbb{N}$ such that $(T - I)^r$ is the zero linear transformation, where I is the identity map on $P_7(x)$

Q.23 For $\alpha \in \mathbb{R}$, let $y_\alpha(x)$ be the solution of the differential equation

$$\frac{dy}{dx} + 2y = \frac{1}{1+x^2} \quad \text{for } x \in \mathbb{R}$$

satisfying $y(0) = \alpha$. Then, which one of the following is TRUE?

- (A) $\lim_{x \rightarrow \infty} y_\alpha(x) = 0$ for every $\alpha \in \mathbb{R}$
- (B) $\lim_{x \rightarrow \infty} y_\alpha(x) = 1$ for every $\alpha \in \mathbb{R}$
- (C) There exists an $\alpha \in \mathbb{R}$ such that $\lim_{x \rightarrow \infty} y_\alpha(x)$ exists but its value is different from 0 and 1
- (D) There is an $\alpha \in \mathbb{R}$ for which $\lim_{x \rightarrow \infty} y_\alpha(x)$ does not exist

Q.24 Consider the following two statements.

P: There exist functions $f: \mathbb{R} \rightarrow \mathbb{R}$, $g: \mathbb{R} \rightarrow \mathbb{R}$ such that f is continuous at $x = 1$ and g is discontinuous at $x = 1$ but $g \circ f$ is continuous at $x = 1$.

Q: There exist functions $f: \mathbb{R} \rightarrow \mathbb{R}$, $g: \mathbb{R} \rightarrow \mathbb{R}$ such that both f and g are discontinuous at $x = 1$ but $g \circ f$ is continuous at $x = 1$.

Which one of the following holds?

- (A) Both P and Q are true
- (B) Both P and Q are false
- (C) P is true but Q is false
- (D) P is false but Q is true

Q.25 Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$f(x) = \frac{(x^2 + 1)^2}{x^4 + x^2 + 1} \quad \text{for } x \in \mathbb{R}.$$

Then, which one of the following is TRUE?

- (A) f has exactly two points of local maxima and exactly three points of local minima
- (B) f has exactly three points of local maxima and exactly two points of local minima
- (C) f has exactly one point of local maximum and exactly two points of local minima
- (D) f has exactly two points of local maxima and exactly one point of local minimum

Q.26 Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a solution of the differential equation

$$\frac{d^2y}{dx^2} - 2 \frac{dy}{dx} + y = 2e^x \text{ for } x \in \mathbb{R}.$$

Consider the following statements.

P: If $f(x) > 0$ for all $x \in \mathbb{R}$, then $f'(x) > 0$ for all $x \in \mathbb{R}$.

Q: If $f'(x) > 0$ for all $x \in \mathbb{R}$, then $f(x) > 0$ for all $x \in \mathbb{R}$.

Then, which one of the following holds?

- (A) P is true but Q is false
- (B) P is false but Q is true
- (C) Both P and Q are true
- (D) Both P and Q are false

Q.27 For $a > b > 0$, consider

$$D = \left\{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq a^2 \text{ and } x^2 + y^2 \geq b^2 \right\}.$$

Then, the surface area of the boundary of the solid D is

- (A) $4\pi(a + b)\sqrt{a^2 - b^2}$
- (B) $4\pi(a^2 - b\sqrt{a^2 - b^2})$
- (C) $4\pi(a - b)\sqrt{a^2 - b^2}$
- (D) $4\pi(a^2 + b\sqrt{a^2 - b^2})$

Q.28 For $n \geq 3$, let a regular n -sided polygon P_n be circumscribed by a circle of radius R_n and let r_n be the radius of the circle inscribed in P_n . Then

$$\lim_{n \rightarrow \infty} \left(\frac{R_n}{r_n} \right)^{n^2}$$

equals

(A) $e^{(\pi^2)}$

(B) $e^{\left(\frac{\pi^2}{2}\right)}$

(C) $e^{\left(\frac{\pi^2}{3}\right)}$

(D) $e^{(2\pi^2)}$

Q.29 Let L_1 denote the line $y = 3x + 2$ and L_2 denote the line $y = 4x + 3$. Suppose that $f: \mathbb{R} \rightarrow \mathbb{R}$ is a four times continuously differentiable function such that the line L_1 intersects the curve $y = f(x)$ at exactly three distinct points and the line L_2 intersects the curve $y = f(x)$ at exactly four distinct points. Then, which one of the following is TRUE?

- (A) $\frac{df}{dx}$ does not attain the value 3 on \mathbb{R}
- (B) $\frac{d^2f}{dx^2}$ vanishes at most once on \mathbb{R}
- (C) $\frac{d^3f}{dx^3}$ vanishes at least once on \mathbb{R}
- (D) $\frac{df}{dx}$ does not attain the value $\frac{7}{2}$ on \mathbb{R}

Q.30 Define the function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$f(x, y) = 12xy e^{-(2x+3y-2)}.$$

If (a, b) is the point of local maximum of f , then $f(a, b)$ equals

- (A) 2
- (B) 6
- (C) 12
- (D) 0

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Section B: Q.31 – Q.40 Carry TWO marks each.

Q.31 Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers.

Then, which of the following statements is/are always TRUE?

(A) If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n^2$ converges absolutely

(B) If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n^3$ converges absolutely

(C) If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n^2$ converges

(D) If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n^3$ converges

Q.32 Which of the following statements is/are TRUE?

(A) $\sum_{n=1}^{\infty} n \log\left(1 + \frac{1}{n^3}\right)$ is convergent

(B) $\sum_{n=1}^{\infty} \left(1 - \cos\left(\frac{1}{n}\right)\right) \log n$ is convergent

(C) $\sum_{n=1}^{\infty} n^2 \log\left(1 + \frac{1}{n^3}\right)$ is convergent

(D) $\sum_{n=1}^{\infty} \left(1 - \cos\left(\frac{1}{\sqrt{n}}\right)\right) \log n$ is convergent

Q.33 Which of the following statements is/are TRUE?

- (A) The additive group of real numbers is isomorphic to the multiplicative group of positive real numbers
- (B) The multiplicative group of nonzero real numbers is isomorphic to the multiplicative group of nonzero complex numbers
- (C) The additive group of real numbers is isomorphic to the multiplicative group of nonzero complex numbers
- (D) The additive group of real numbers is isomorphic to the additive group of rational numbers

Q.34 Let $f: (1, \infty) \rightarrow (0, \infty)$ be a continuous function such that for every $n \in \mathbb{N}$, $f(n)$ is the smallest prime factor of n . Then, which of the following options is/are CORRECT?

- (A) $\lim_{x \rightarrow \infty} f(x)$ exists
- (B) $\lim_{x \rightarrow \infty} f(x)$ does not exist
- (C) The set of solutions to the equation $f(x) = 2024$ is finite
- (D) The set of solutions to the equation $f(x) = 2024$ is infinite

Q.35

Let

$$S = \{(x, y) \in \mathbb{R}^2: x > 0, y > 0\},$$

and $f: S \rightarrow \mathbb{R}$ be given by

$$f(x, y) = 2x^2 + 3y^2 - \log x - \frac{1}{6} \log y.$$

Then, which of the following statements is/are TRUE?

- (A) There is a unique point in S at which $f(x, y)$ attains a local maximum
- (B) There is a unique point in S at which $f(x, y)$ attains a local minimum
- (C) For each point $(x_0, y_0) \in S$, the set $\{(x, y) \in S: f(x, y) = f(x_0, y_0)\}$ is bounded
- (D) For each point $(x_0, y_0) \in S$, the set $\{(x, y) \in S: f(x, y) = f(x_0, y_0)\}$ is unbounded

Q.36 The center $Z(G)$ of a group G is defined as

$$Z(G) = \{x \in G : xg = gx \text{ for all } g \in G\}.$$

Let $|G|$ denote the order of G . Then, which of the following statements is/are TRUE for any group G ?

- (A) If G is non-abelian and $Z(G)$ contains more than one element, then the center of the quotient group $G/Z(G)$ contains only one element
- (B) If $|G| \geq 2$, then there exists a non-trivial homomorphism from \mathbb{Z} to G
- (C) If $|G| \geq 2$ and G is non-abelian, then there exists a non-identity isomorphism from G to itself
- (D) If $|G| = p^3$, where p is a prime number, then G is necessarily abelian

Q.37 For a matrix M , let $\text{Rowspace}(M)$ denote the linear span of the rows of M and $\text{Colspace}(M)$ denote the linear span of the columns of M . Which of the following hold(s) for all $A, B, C \in M_{10}(\mathbb{R})$ satisfying $A = BC$?

(A) $\text{Rowspace}(A) \subseteq \text{Rowspace}(B)$

(B) $\text{Rowspace}(A) \subseteq \text{Rowspace}(C)$

(C) $\text{Colspace}(A) \subseteq \text{Colspace}(B)$

(D) $\text{Colspace}(A) \subseteq \text{Colspace}(C)$

Q.38 Define $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$ as follows

$$f(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m} (m!)^2} \text{ and } g(x) = \frac{x}{2} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m} (m+1)! m!} \text{ for } x \in \mathbb{R}.$$

Let $x_1, x_2, x_3, x_4 \in \mathbb{R}$ be such that $0 < x_1 < x_2$, $0 < x_3 < x_4$,

$$f(x_1) = f(x_2) = 0, \quad f(x) \neq 0 \text{ when } x_1 < x < x_2,$$

$$g(x_3) = g(x_4) = 0 \quad \text{and } g(x) \neq 0 \text{ when } x_3 < x < x_4.$$

Then, which of the following statements is/are TRUE?

- (A) The function f does not vanish anywhere in the interval (x_3, x_4)
- (B) The function f vanishes exactly once in the interval (x_3, x_4)
- (C) The function g does not vanish anywhere in the interval (x_1, x_2)
- (D) The function g vanishes exactly once in the interval (x_1, x_2)

Q.39 For $0 < \alpha < 4$, define the sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers as follows:

$$x_1 = \alpha \text{ and } x_{n+1} + 2 = -x_n(x_n - 4) \text{ for } n \in \mathbb{N}.$$

Which of the following is/are TRUE?

- (A) $\{x_n\}_{n=1}^{\infty}$ converges for at least three distinct values of $\alpha \in (0,1)$
- (B) $\{x_n\}_{n=1}^{\infty}$ converges for at least three distinct values of $\alpha \in (1,2)$
- (C) $\{x_n\}_{n=1}^{\infty}$ converges for at least three distinct values of $\alpha \in (2,3)$
- (D) $\{x_n\}_{n=1}^{\infty}$ converges for at least three distinct values of $\alpha \in (3,4)$

Q.40 Consider

$$G = \{m + n\sqrt{2} : m, n \in \mathbb{Z}\}$$

as a subgroup of the additive group \mathbb{R} .

Which of the following statements is/are TRUE?

- (A) G is a cyclic subgroup of \mathbb{R} under addition
- (B) $G \cap I$ is non-empty for every non-empty open interval $I \subseteq \mathbb{R}$
- (C) G is a closed subset of \mathbb{R}
- (D) G is isomorphic to the group $\mathbb{Z} \times \mathbb{Z}$, where the group operation in $\mathbb{Z} \times \mathbb{Z}$ is defined by $(m_1, n_1) + (m_2, n_2) = (m_1 + m_2, n_1 + n_2)$

Section C: Q.41 – Q.50 Carry ONE mark each.

Q.41 The area of the region

$$R = \left\{ (x, y) \in \mathbb{R}^2: 0 \leq x \leq 1, 0 \leq y \leq 1 \text{ and } \frac{1}{4} \leq xy \leq \frac{1}{2} \right\}$$

is _____ (rounded off to two decimal places).

Q.42 Let $y: \mathbb{R} \rightarrow \mathbb{R}$ be the solution to the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 1$$

satisfying $y(0) = 0$ and $y'(0) = 1$.

Then, $\lim_{x \rightarrow \infty} y(x)$ equals _____ (rounded off to two decimal places).

Q.43 For $\alpha > 0$, let $y_\alpha(x)$ be the solution to the differential equation

$$2\frac{d^2y}{dx^2} - \frac{dy}{dx} - y = 0$$

satisfying the conditions

$$y(0) = 1, \quad y'(0) = \alpha.$$

Then, the smallest value of α for which $y_\alpha(x)$ has no critical points in \mathbb{R} equals

_____ (rounded off to the nearest integer).

Q.44 Consider the 4×4 matrix

$$M = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}.$$

If $a_{i,j}$ denotes the (i,j) th entry of M^{-1} , then $a_{4,1}$ equals _____ (rounded off to two decimal places).

Q.45 Let $P_{12}(x)$ be the real vector space of polynomials in the variable x with real coefficients and having degree at most 12, together with the zero polynomial.

Define

$$V = \left\{ f \in P_{12}(x) : f(-x) = f(x) \text{ for all } x \in \mathbb{R} \text{ and } f(2024) = 0 \right\}.$$

Then, the dimension of V is _____

Q.46 Let

$$S = \left\{ f: \mathbb{R} \rightarrow \mathbb{R} : f \text{ is a polynomial and } f(f(x)) = (f(x))^{2024} \text{ for } x \in \mathbb{R} \right\}.$$

Then, the number of elements in S is _____

Q.47 Let $a_1 = 1, b_1 = 2$ and $c_1 = 3$. Consider the convergent sequences

$$\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty} \text{ and } \{c_n\}_{n=1}^{\infty}$$

defined as follows:

$$a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \frac{b_n + c_n}{2} \text{ and } c_{n+1} = \frac{c_n + a_n}{2} \text{ for } n \geq 1.$$

Then,

$$\sum_{n=1}^{\infty} b_n c_n (a_{n+1} - a_n) + \sum_{n=1}^{\infty} (b_{n+1} c_{n+1} - b_n c_n) a_{n+1}$$

equals _____ (rounded off to two decimal places)

Q.48 Let

$$S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 4, (x - 1)^2 + y^2 \leq 1, z \geq 0\}.$$

Then, the surface area of S equals _____ (rounded off to two decimal places).

Q.49 Let $P_7(x)$ be the real vector space of polynomials in x with degree at most 7, together with the zero polynomial. For $r = 1, 2, \dots, 7$, define

$$s_r(x) = x(x-1)\cdots(x-(r-1)) \text{ and } s_0(x) = 1.$$

Consider the fact that $B = \{s_0(x), s_1(x), \dots, s_7(x)\}$ is a basis of $P_7(x)$.

If

$$x^5 = \sum_{k=0}^7 \alpha_{5,k} s_k(x),$$

where $\alpha_{5,k} \in \mathbb{R}$, then $\alpha_{5,2}$ equals _____ (rounded off to two decimal places)

Q.50 Let

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 2 & 0 & 0 & 0 & -4 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 3 \\ 0 & 0 & 0 & 2 & 2 \end{pmatrix}.$$

If $p(x)$ is the characteristic polynomial of M , then $p(2) - 1$ equals _____

Section C: Q.51 – Q.60 Carry TWO marks each.

Q.51 For $\alpha \in (-2\pi, 0)$, consider the differential equation

$$x^2 \frac{d^2y}{dx^2} + \alpha x \frac{dy}{dx} + y = 0 \quad \text{for } x > 0.$$

Let D be the set of all $\alpha \in (-2\pi, 0)$ for which all corresponding real solutions to the above differential equation approach zero as $x \rightarrow 0^+$. Then, the number of elements in $D \cap \mathbb{Z}$ equals _____

Q.52 The value of

$$\lim_{t \rightarrow \infty} \left(\left(\log \left(t^2 + \frac{1}{t^2} \right) \right)^{-1} \int_1^{\pi t} \frac{\sin^2 5x}{x} dx \right)$$

equals _____ (rounded off to two decimal places).

- Q.53 Let T be the planar region enclosed by the square with vertices at the points $(0,1)$, $(1,0)$, $(0,-1)$ and $(-1,0)$. Then, the value of

$$\iint_T \left(\cos(\pi(x-y)) - \cos(\pi(x+y)) \right)^2 dx dy$$

equals _____ (rounded off to two decimal places).

- Q.54 Let

$$S = \{(x, y, z) \in \mathbb{R}^3: x^2 + y^2 + z^2 < 1\}.$$

Then, the value of

$$\frac{1}{\pi} \iiint_S \left((x - 2y + z)^2 + (2x - y - z)^2 + (x - y + 2z)^2 \right) dx dy dz$$

equals _____ (rounded off to two decimal places).

- Q.55 For $n \in \mathbb{N}$, if

$$a_n = \frac{1}{n^3 + 1} + \frac{2^2}{n^3 + 2} + \dots + \frac{n^2}{n^3 + n}$$

then the sequence $\{a_n\}_{n=1}^{\infty}$ converges to _____ (rounded off to two decimal places)

Q.56 Consider the function $f: \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = x^3 - 4x^2 + 4x - 6$.

For $c \in \mathbb{R}$, let

$$S(c) = \left\{ x \in \mathbb{R} : f(x) = c \right\}$$

and $|S(c)|$ denote the number of elements in $S(c)$. Then, the value of

$$|S(-7)| + |S(-5)| + |S(3)|$$

equals _____

Q.57 Let $c > 0$ be such that

$$\int_0^c e^{s^2} ds = 3$$

Then, the value of

$$\int_0^c \left(\int_x^c e^{x^2+y^2} dy \right) dx$$

equals _____ (rounded off to one decimal place).

- Q.58 For $k \in \mathbb{N}$, let $0 = t_0 < t_1 < \dots < t_k < t_{k+1} = 1$. A function $f: [0,1] \rightarrow \mathbb{R}$ is said to be piecewise linear with nodes t_1, \dots, t_k , if for each $j = 1, 2, \dots, k+1$, there exist $a_j \in \mathbb{R}$, $b_j \in \mathbb{R}$ such that

$$f(t) = a_j + b_j t \text{ for } t_{j-1} < t < t_j.$$

Let V be the real vector space of all real valued continuous piecewise linear functions on $[0,1]$ with nodes $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}$. Then, the dimension of V equals

- Q.59 For $n \in \mathbb{N}$, let

$$a_n = \frac{1}{n^{n-1}} \sum_{k=0}^n \frac{n!}{k!(n-k)!} \frac{n^k}{k+1}$$

and $\beta = \lim_{n \rightarrow \infty} a_n$. Then, the value of $\log \beta$ equals _____ (rounded off to two decimal places).

- Q.60 Define the function $f: (-1,1) \rightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ by

$$f(x) = \sin^{-1} x$$

Let a_6 denote the coefficient of x^6 in the Taylor series of $(f(x))^2$ about

$x = 0$. Then, the value of $9a_6$ equals _____ (rounded off to two decimal places).