

**JAM 2006**

**MATHEMATICS TEST PAPER**

**NOTATIONS USED**

- $\mathbb{R}$  : The set of all real numbers  
 $\mathbb{Z}$  : The set of all integers

**IMPORTANT NOTE FOR CANDIDATES**

**Objective Part:**

Attempt **ALL** the objective questions (Questions 1-15). Each of these questions carries six marks. Each incorrect answer carries minus two. Write the answers to the objective questions in the Answer Table for Objective Questions provided on page 7 only.

**Subjective Part:**

Attempt **ALL** subjective questions (Questions 16-29). Each of these questions carries fifteen marks.

1.  $\lim_{n \rightarrow \infty} \frac{2^{n+1} + 3^{n+1}}{2^n + 3^n}$  equals
  - (A) 3
  - (B) 2
  - (C) 1
  - (D) 0
  
2. Let  $f(x) = (x - 2)^{17}(x + 5)^{24}$ . Then
  - (A)  $f$  does not have a critical point at 2
  - (B)  $f$  has a minimum at 2
  - (C)  $f$  has a maximum at 2
  - (D)  $f$  has neither a minimum nor a maximum at 2
  
3. Let  $f(x, y) = x^5 y^2 \tan^{-1}\left(\frac{y}{x}\right)$ . Then  $x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y}$  equals
  - (A)  $2f$
  - (B)  $3f$
  - (C)  $5f$
  - (D)  $7f$
  
4. Let  $G$  be the set of all irrational numbers. The interior and the closure of  $G$  are denoted by  $G^0$  and  $\overline{G}$ , respectively. Then
  - (A)  $G^0 = \phi$ ,  $\overline{G} = G$
  - (B)  $G^0 = \mathbb{R}$ ,  $\overline{G} = \mathbb{R}$
  - (C)  $G^0 = \phi$ ,  $\overline{G} = \mathbb{R}$
  - (D)  $G^0 = G$ ,  $\overline{G} = \mathbb{R}$

5. Let  $f(x) = \int_{\sin x}^{\cos x} e^{-t^2} dt$ . Then  $f'(\pi/4)$  equals

- (A)  $\sqrt{1/e}$
- (B)  $-\sqrt{2/e}$
- (C)  $\sqrt{2/e}$
- (D)  $-\sqrt{1/e}$

6. Let  $C$  be the circle  $x^2 + y^2 = 1$  taken in the anti-clockwise sense. Then the value of the integral

$$\int_C [(2xy^3 + y)dx + (3x^2y^2 + 2x)dy]$$

equals

- (A) 1
- (B)  $\pi/2$
- (C)  $\pi$
- (D) 0

7. Let  $r$  be the distance of a point  $P(x, y, z)$  from the origin  $O$ . Then  $\nabla r$  is a vector

- (A) orthogonal to  $\overrightarrow{OP}$
- (B) normal to the level surface of  $r$  at  $P$
- (C) normal to the surface of revolution generated by  $OP$  about x-axis
- (D) normal to the surface of revolution generated by  $OP$  about y-axis

8. Let  $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be defined by

$$T(x_1, x_2, x_3) = (x_1 - x_2, x_1 - x_2, 0).$$

If  $N(T)$  and  $R(T)$  denote the null space and the range space of  $T$  respectively, then

- (A)  $\dim N(T) = 2$
- (B)  $\dim R(T) = 2$
- (C)  $R(T) = N(T)$
- (D)  $N(T) \subset R(T)$

9. Let  $S$  be a closed surface for which  $\iint_S \vec{r} \cdot \hat{n} d\sigma = 1$ . Then the volume enclosed by the

surface is

- (A) 1
- (B)  $1/3$
- (C)  $2/3$
- (D) 3

10. If  $(c_1 + c_2 \ln x)/x$  is the general solution of the differential equation

$$x^2 \frac{d^2 y}{dx^2} + kx \frac{dy}{dx} + y = 0, \quad x > 0,$$

then  $k$  equals

- (A) 3  
(B) -3  
(C) 2  
(D) -1
11. If  $A$  and  $B$  are  $3 \times 3$  real matrices such that  $\text{rank}(AB) = 1$ , then  $\text{rank}(BA)$  **cannot** be
- (A) 0  
(B) 1  
(C) 2  
(D) 3
12. The differential equation representing the family of circles touching  $y$ -axis at the origin is
- (A) linear and of first order  
(B) linear and of second order  
(C) nonlinear and of first order  
(D) nonlinear and of second order
13. Let  $G$  be a group of order 7 and  $\phi(x) = x^4, x \in G$ . Then  $\phi$  is
- (A) not one – one  
(B) not onto  
(C) not a homomorphism  
(D) one – one, onto and a homomorphism
14. Let  $R$  be the ring of all  $2 \times 2$  matrices with integer entries. Which of the following subsets of  $R$  is an integral domain?
- (A)  $\left\{ \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} : x, y \in \mathbf{Z} \right\}$   
(B)  $\left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} : x, y \in \mathbf{Z} \right\}$   
(C)  $\left\{ \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} : x \in \mathbf{Z} \right\}$   
(D)  $\left\{ \begin{pmatrix} x & y \\ y & z \end{pmatrix} : x, y, z \in \mathbf{Z} \right\}$

15. Let  $f_n(x) = n \sin^{2n+1} x \cos x$ . Then the value of

$$\lim_{n \rightarrow \infty} \int_0^{\pi/2} f_n(x) dx - \int_0^{\pi/2} \left( \lim_{n \rightarrow \infty} f_n(x) \right) dx$$

is

- (A)  $1/2$
- (B)  $0$
- (C)  $-1/2$
- (D)  $-\infty$

16. (a) Test the convergence of the series

$$\sum_{n=1}^{\infty} \frac{n^n}{n! 3^n} \quad (6)$$

(b) Show that

$$\ln(1 + \cos x) \leq \ln 2 - \frac{x^2}{4}$$

for  $0 \leq x \leq \pi/2$ . (9)

17. Find the critical points of the function

$$f(x, y) = x^3 + y^2 - 12x - 6y + 40.$$

Test each of these for maximum and minimum. (15)

18. (a) Evaluate  $\iint_R x e^{y^2} dx dy$ , where  $R$  is the region bounded by the lines  $x = 0$ ,  $y = 1$  and the parabola  $y = x^2$ . (6)

(b) Find the volume of the solid bounded above by the surface  $z = 1 - x^2 - y^2$  and below by the plane  $z = 0$ . (9)

19. Evaluate the surface integral

$$\iint_S x(12y - y^4 + z^2) d\sigma,$$

where the surface  $S$  is represented in the form  $z = y^2$ ,  $0 \leq x \leq 1$ ,  $0 \leq y \leq 1$ . (15)

20. Using the change of variables, evaluate  $\iint_R xy dx dy$ , where the region  $R$  is bounded by the curves  $xy = 1$ ,  $xy = 3$ ,  $y = 3x$  and  $y = 5x$  in the first quadrant. (15)

21. (a) Let  $u$  and  $v$  be the eigenvectors of  $A$  corresponding to the eigenvalues 1 and 3 respectively. Prove that  $u + v$  is not an eigenvector of  $A$ . (6)

(b) Let  $A$  and  $B$  be real matrices such that the sum of each row of  $A$  is 1 and the sum of each row of  $B$  is 2. Then show that 2 is an eigenvalue of  $AB$ . (9)

22. Suppose  $W_1$  and  $W_2$  are subspaces of  $\mathbb{R}^4$  spanned by  $\{(1,2,3,4), (2,1,1,2)\}$  and  $\{(1,0,1,0), (3,0,1,0)\}$  respectively. Find a basis of  $W_1 \cap W_2$ . Also find a basis of  $W_1 + W_2$  containing  $\{(1,0,1,0), (3,0,1,0)\}$ . (15)

23. Determine  $y_0$  such that the solution of the differential equation

$$y' - y = 1 - e^{-x}, \quad y(0) = y_0$$

has a finite limit as  $x \rightarrow \infty$ . (15)

24. Let  $\phi(x, y, z) = e^x \sin y$ . Evaluate the surface integral  $\iint_S \frac{\partial \phi}{\partial n} d\sigma$ , where  $S$  is the surface of the cube  $0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1$  and  $\frac{\partial \phi}{\partial n}$  is the directional derivative of  $\phi$  in the direction of the unit outward normal to  $S$ . Verify the divergence theorem. (15)

25. Let  $y = f(x)$  be a twice continuously differentiable function on  $(0, \infty)$  satisfying

$$f(1) = 1 \text{ and } f'(x) = \frac{1}{2} f\left(\frac{1}{x}\right), \quad x > 0.$$

Form the second order differential equation satisfied by  $y = f(x)$ , and obtain its solution satisfying the given conditions. (15)

26. Let  $G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbf{Z} \right\}$  be the group under matrix addition and  $H$  be the subgroup of  $G$  consisting of matrices with even entries. Find the order of the quotient group  $G/H$ . (15)

27. Let

$$f(x) = \begin{cases} x^2 & 0 \leq x \leq 1 \\ \sqrt{x} & x > 1. \end{cases}$$

Show that  $f$  is uniformly continuous on  $[0, \infty)$ . (15)

28. Find  $M_n = \max_{x \geq 0} \left\{ \frac{x}{n(1+nx^3)} \right\}$ , and hence prove that the series

$$\sum_{n=1}^{\infty} \frac{x}{n(1+nx^3)}$$

is uniformly convergent on  $[0, \infty)$ . (15)

29. Let  $R$  be the ring of polynomials with real coefficients under polynomial addition and polynomial multiplication. Suppose

$$I = \{ p \in R : \text{sum of the coefficients of } p \text{ is zero} \}.$$

Prove that  $I$  is a maximal ideal of  $R$ . (15)